International Journal of Theoretical Physics, Vol. 25, No. 6, 1986

Equivalence of the Degrees of Freedom in a Unified Gravitational Theory

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Received December 10, 1985

A discussion of the nonuniqueness of physical laws and their invariance groups is illustrated by the construction of a physical theory (presented earlier) in which the law of motion of structureless and spinning particles is unified in the geometry of the manifold of the de Sitter group $SO(3, 2)$. The theory has the structure of a non-Abelian Kaluza-Klein theory with very special properties resulting from the topology and noncompactness of the groups. The physical interpretation of the field equation is discussed. The physical requirement of equivalence of the interaction of spinning and orbiting systems, generally unconsidered in related theories, is here taken into account by the structure of the theory. The possibility of deviations from predictions of general relativity exists. Generalizations of the theoretical structure to higher dimensional groups are outlined and open the possibility for observations.

1. INTRODUCTION

The occurrence of invariance groups in physics is a necessary consequence of simplifying assumptions that we are compelled to make in describing nature. We choose a set of phenomena that appear to be as little as possible interfered with by the rest of the world and declare them as fundamental and truly independent of the rest of the world. These phenomena usually imply a (continuous) multitude of possible configurations of the system considered relative to the rest of the world, and our assumptions imply that whichever of these configurations is tacitly assumed will not change the physical situation of any of these two parts of the universe by even a bit. The invariance group appears in the description of the relations between the different possible configurations. If a deviation occurs from the prescribed law, a culprit has to be found whose interference

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0020-7748/86/0600-0599$05.00/0

causes the deviation; but even the latter is expected to obey the law. The necessary shortcomings of such a procedure make the importance of infinite-dimensional groups (e.g., those of gauge theories) for physics understandable. These are able to remedy the situation when a law, which seems to be locally well obeyed at different spots, would lead on a global scale to irreconcilable consequences. A universal culprit has to be found for this irreconcilability who is to interfere between infinitely many such spots. In this way the description of nature progresses to a more general form. It is remarkable that constructions that involve an infinity of entities—in particular the continuum—and which clearly have no place in an objective external physical world which is independent of our perception had to be introduced in order to achieve progress in the description of this world.[2]

The procedure described above to arrive at fundamental laws of physics is clearly not unique, and alternative laws may be proposed the adoption of which will prove advantageous if they lead to a more economic or more general description. The invariance group will have to be modified in accord with the choice of the fundamental law.

The outlined method, which no doubt does not just reflect the views held by many founders of physics, can nevertheless be illustrated by historical development and suggests a host of possibilities for (transient) approaches in the future.[3]

After gaining a better insight into the invariance properties of the physics of space-time, exploration of the inner degrees of freedom of matter and its symmetries came to the forefront.

Einstein gave the law of motion its global form, which geometrized the gravitational interaction. Two outstanding attempts were then made also to include the electromagnetic interaction in this geometrization:

The first gauge theory of the units of length generalizing the Christoffel connection of Riemannian geometry Weyl (1922) and the five-dimensional metric theory of Kaluza (1921) and Klein (1926). Both approaches were then formally generalized from the one-dimensional invariance group of electrodynamics to non-Abelian groups (Yang and Mills, 1954; De Witt,

[2]Such a paradoxical development for the sake of progress seems to occur again at the present stage, when the space-time continuum is divided into discrete elements in the lattice gauge theories. Although we do lack knowledge of how to find fundamental discrete elements in continuum physics, we have all indications that space-time has a statistical background and is not a good candidate (Halpern, 1973).

[3]Buddhist philosophy sees the cause of all suffering in the futile clinging to features of reality vainly categorized by the mind. Projected on the colorless world of the physicist, this truth may state that the cause for suffering of physicists is the clinging to certain invariance groups. This may be illustrated by P. Lenard and his followers clinging to the Galilei group in their futile endeavor to present physics as "German" and non-Einsteinian.

1964). This generalization allows one to formally include the inner degrees of freedom of matter into the geometrization.

Among these inner degrees of freedom of matter, the spin of elementary particles has the peculiar property to be convertible into angular momentum—a conventional dynamical variable in space-time. This property should make its geometrization of interest beyond all formal aspects. An early generalization of general relativity led to the idea of identifying spin with torsion (Cartan, 1923/24; Hehl et al., 1976; Trautmann, 1972). This idea was later modified to a gauge theory of the Poincaré group, which results in a connection with torsion; the latter is related to the elementary particle spin, not, however, to angular momentum. A gauge theory of the group of tetrade rotations (the proper Lorentz group) has been considered by the present author (Halpern and Miketinac, 1970) in order to make a tetrade formulation of gravitational theory, suggested by Moeller, unique and explain *CP* nonconservation. A gauge formalism of the unified spin rotations and electromagnetic gauge transformations of the Dirac equation was considered in a later paper, which also touched upon the possibility of a unified dual charge (Halpern, 1977a).

A modification of the general theory of relativity where the asymptotic symmetry is that of the de Sitter group was already contemplated by Lubkin (1972) (see also Halpern, 1977b).

A gauge theory of gravitation with the de Sitter group as gauge group was also suggested by Hsu (1979). The present author extended the gauge formalism associated with the Dirac equation to Dirac's de Sitter covariant spinor equation (Dirac, 1935, 1936). This resulted in a general gauge formalism of the de Sitter group from which I, however, intended to exclude torsion (Halpern, 1977a). Considering the Poincaré group, which is a contraction of the de Sitter group, only as an approximation which may suffer the fate of other contractions of pseudo-orthogonal groups—to be replaced by the full group—I speculated that orbits of all one-dimensional subgroups of the simple de Sitter group have to appear on an equal footing in the theory. The law of motion in a truly de Sitter covariant theory has thus to be generalized. One possibility to achieve this appeared to be the unification of the law of motion of structureless and spinning particles. Such a unification should also take account of the author's criticism of Cartan's theory and the Poincaré gauge theory with their separation of spin and angular momentum. Following the considerations outlined at the beginning of this section, one can conclude that there is *a priori* no reason to prefer the Poincaré group over other invariance groups, just as there is no reason to assume that the metric of space ought to be flat. The simple de Sitter group is in many respects more appealing mathematically and even to physicists who believe in an interrelation of all phenomena in nature.

The modification outlined must be performed completely if it is to be anything more than an artifice to arrive at conventional results in a limit. With the translation subgroup of the Poincaré group also the flat space background of the theory should be modified to the space of constant curvature of the de Sitter universe. The physical laws are then best formulated on the group of which the universe is the coset space G/H, with H the subgroup $SO(3,1)$. Every representation of the Lie algebra of G is composed of the representations by functions on the group manifold. These thus also include representations of the subgroup that correspond to half-integer spin (Bopp and Haag 1950). Based on the ideas outlined at the beginning of this section, I have phrased the modification of the law of inertia as follows: "A body moves along the timelike orbit of a (one-dimensional) subgroup of the de Sitter group on the de Sitter universe unless interfered with" (Halpern, 1984a).

These orbits comprise, besides the timelike geodesics, also the motion of particles with spin. The generalization is not spectacular in case of the symmetric space of the de Sitter universe, where spin does not really affect the motion. The modifications only become observable when the law is generalized to the inhomogeneous metric of localized sources.

The manifold of a semisimple group has a natural metric γ which is a measure of the noncommutability of its generators. The metric g of the de Sitter universe is then induced in our case by the natural projection of γ on G/H. The metric γ always fulfills Einstein's equations with a cosmological member. This makes γ and G special solutions of a generalized Kaluza-Klein theory. Such theories describe internal degrees of freedom of elementary particles with a metric in a higher dimensional Riemannian space. It is of particular interest to describe the spin this way because of its relation to angular momentum. Remarkably, even after Kaluza-Klein theories recently came into fashion, contributions along this line were lacking.[4] This is no doubt due to the great success of the Dirac spinor and the challenging mathematical problems of spinor theories in higher dimensions. It should be stressed, however, that the motivation of the present research is a different one: *trying to arrive at a unified description of physics of space-time and inner degrees of freedom with the help of a symmetry group and its manifold. This is different from collecting the known properties of physics in space-time and the inner degrees of freedom and generalizing these to spaces of higher dimensions.*

Given this difference, e.g., a supersymmetric generalization of the present group covariant theory would not make much sense.

[4]The work of Fukuyama (1982) apparently follows a reasoning close to the present one. In this interesting work spin is also suggested to be the dual charge of the gravitational field.

Spin has, apart from its kinship with angular momentum, other fundamental properties, e.g., its relation to statistics. Instead of inserting these, a way should be sought to relate them in the theory. In Halpern (1984a), it was mentioned that there are some indications (certainly not yet more than just indications) that because the universe of $SO(3,2)/SO(3,1)$ is closed in time, these properties may have their fundamentals in the theory. This problem is not yet ripe for discussion. The theory is also not yet in a stage where insertions of empirically determined constants have to be made. All relations on the group manifolds are purely geometrical; there is the hope that extended theories with a higher dimensional group G will yield further relations of this kind (see Section 6). How are the physical dimensions determined? A previous paper (Halpern, 1984b) introduced the dimension of length on the group manifold G itself (a measure of the noncommutability of the generators). The present treatment leaves the group manifold dimensionless. The physical dimension of length is introduced only with the projection on the factor space G/H, which becomes then the space-time of the de Sitter universe. Choosing as unit length the radius of this universe and considering Newton's gravitational constant as dimensionless, action acquires the dimension of length squared. The length determined by the quantum of action (Planck length) would, according to Dirac's large numbers hypothesis (Dirac, 1979), vary relative to the radius (or the age) of an expanding universe. A theory with higher dimensional group G can describe such a situation and obtain the large ratio at present time geometrically. The problem is not discussed here further.

The Kaluza-Klein formulation on the group manifold was suggested at a time when there was not yet much interest in such structures (Halpern, 1979). A large variety of sophisticated and adaptable mathematical models to describe inner degrees of freedom in higher dimensions have been suggested more recently (e.g., Jadczyk, 1984). Should one stick to the present rigid construction with disturbing features, as a noncompact group H, which in the Yang-Mills formulation gives rise to nondefinite energy of the gauge field?

As a gauge theory the author has considered such a theory much earlier. The Kaluza-Klein formulation on the group manifold which emerged from the physical considerations outlined acquires, however, a completely different character due to the topology of the group manifold and the boundary conditions. Some of the differences show up in the vertical part of the Einstein equation, which appear as subsidiary conditions in the gauge theory; other differences result from the boundary conditions, which are unusual for a Yang-Mills theory. The linear connection in case of a general metric γ consists of a Christoffel part and a torsion part (contortion), which is a tensor field. New exact solutions of the field equations have not yet been obtained. The form of the equations and of the boundary conditions show, however, that physical effects modifying general relativity are due to the curvature of the full connection, rather than the torsion. The mentioned conditions keep this curvature in many cases small. The vacuum solutions of general relativity with cosmological members are also solutions of the present theory. They are associated with a large torsion tensor, which compensates the metric part in the curvature tensor so that the latter remains of cosmological smallness. The spin of elementary particles interacts in the theory with the full curvature and not with the Riemannian curvature. Spin effects predicted by general relativity are therefore expected to be in general much smaller in the present theory due to the compensating effects of torsion.

A major motivation for the construction of the theory is the physical requirement of the equivalent interaction of spin and angular momentum. A spinning particle should at larger distances give rise to similar fields as, e.g., two structureless bodies in Keplerian motion of equal angular momentum. This requirement seems not to have been considered in other theories. The angular momentum of the "spin term" of the energy-momentum tensor is in the present theory equal to the torsion current. The mixed horizontal-vertical part of the Einstein equations also show that torsion has an additional source derived from the Ricci tensor; it is, however, not established that this source fulfills the equivalence requirement if only structureless bodies exist. If the equivalence holds, the theory predicts deviations from relativistic motion of macroscopic spinning bodies that may become observable.

It should be stressed that the theory is still in a preliminary stage. It lacks detailed models of spinning bodies. The mathematical foundations have been worked out, but the author feels this mathematical structure is only a first step in the exploration of the challenging possibilities that exist for the role of spin interactions in the universe. The next necessary step is seen in the modification of the rigid boundary conditions to a dependence on the sources in the universe. The vertical components of Einstein's equations in ten dimensions fulfill only a part of this task.

2. THE GROUP MANIFOLD AND THE UNIVERSE

We want to carry out the formulation of physical laws in terms of a semisimple invariance group G as completely as possible. The natural background for the mathematical formulation is the group manifold of G. Observations can only be made in space-time. The space-time manifold is described in terms of the left coset space G/H with respect to a semisimple subgroup H of G.

The simplest case that we are considering is that of the de Sitter group $SO(4,1)$ or preferably, because of its local causality properties (Philips and Wigner, 1968; Segal, 1976), the anti-de Sitter group $SO(3,2)$ for G and the Lorentz group $SO(3,1)$ for H. The resulting de Sitter and anti-de Sitter universes are spaces of constant curvature (generalized spheres with one or two imaginary coordinates). The one is spatially closed and open in timelike directions, the other is spatially open and closed in timelike directions. The hope is that the modified invariance group generalizes the laws of motion to incorporate that of elementary particles with spin and result on the macroscopic level in a generalization of the theory of gravitation. A related formulation can be applied to larger pseudo-orthogonal groups to include other internal degrees of freedom.

The group G may be considered in a natural way as principal fibre bundle $P(G, \pi, H, G/H)$ over G/H, where π is the natural projection $G \to G/H$.

The Cartan-Killing metric γ on G

$$\gamma_{RS} = c^U{}_{RV} c^V{}_{SU} \tag{1}$$

induces a metric g on the base space G/H which is conformally flat (metric of the de Sitter universe). G is invariant under the natural left action of G on G/H. The bundle of orthonormal frames over the base space also has H as its group and is equivalent to P.

The Lie algebra of G admits a Cartan decomposition into the direct sum of the Lie algebra $\mathfrak{h}$ of H and its orthogonal to the Cartan-Killing inner product,

$$\begin{gathered} \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}, \qquad \mathfrak{k} = \mathfrak{h}^{\perp} \\ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}, \qquad [\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{k}, \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \end{gathered} \tag{2}$$

Considered as a left invariant distribution on G, $\mathfrak{k}$ determines a left invariant connection as a principal H-bundle. This is equivalent to the connection on the orthonormal frame bundle of G/H associated with the base metric g. The horizontal subspaces are orthogonal to the vertical subspaces with respect to the Cartan-Killing metric γ on G. The inner product of the vertical subspaces differs from the Cartan-Killing inner product of G itself only by a conformal constant, which is $3/2$ in our example.

The Cartan-Killing metric on H is a solution of the vacuum Einstein equations with a cosmological constant which equals 1 in our example. On the base manifold the induced metric of this solution is that of the de Sitter universe. The group manifold with its natural metric can thus be identified with the vacuum solution of a particular Kaluza-Klein theory with the Lorentz group as its gauge group.

The introduction of the dimension of length is required to relate this geometrical construction to physics. In a previous publication (Halpern, 1984b) the group manifold was endowed with such a dimension; in the present work the dimension of length is only introduced on the factor space G/H on which the metric of G is projected. On the base manifold G/H we keep the coordinates dimensionless and endow the components of the covariant metric tensor with the dimension of length squared.

We shall henceforth denote the dimension of G by r and the dimension of H by $r-k$. In case of the de Sitter group $r=10$ and $k=4$. Capital Latin indices pertain in general to an orthonormal base and lowercase indices to a coordinate base. The Einstein summation convention for letters before and including k in the alphabet runs from 1 to k; for letters after k, inclusive of q, from $r-k$ until r; and that for letters after q runs from 1 to $r=\dim G$. This rule will be applied without further warning.

Locally the group manifold is homeomorphic to $G/H\times H$, so that local coordinates can be introduced on G such that the first coordinates x^e label the points of the base space and the following $r-k$ coordinates label the points on the fibre over it. Vectors on the fibre have then no components in the x^e direction.

We can in such a coordinate system relate the components of the metric tensor g on the base by our dimensional considerations to those on G:

$$g^{ik}=\alpha^{-2}\gamma^{ik} \tag{3}$$

where α is a constant with the dimension of a length. We use in general units of length for which $\alpha=1$, which is of the order of the radius of the de Sitter universe. Smaller units of length entail larger values of α.

3. THE DEGREES OF FREEDOM OF THE FIELDS

We have contemplated in the last section the geometry of the manifold of the de Sitter group G and saw that its natural metric is a solution of Einstein's equations in $r=10$ dimensions. The space-time of the de Sitter universe results from the natural projection on the factor space G/H.

The equations of motions of test bodies are also determined by Einstein's equations. Their one-dimensional solutions are the geodesics of G. The projection of these on space-time are the orbits of G on G/H, which also comprise the geodesics. We try to relate a general orbit to the motion of a particle with the inner degree of freedom of spin. This is indeed possible in the case of the de Sitter metric: The condition has to be imposed on the velocity vector that its horizontal and vertical components commute [in the case of $SO(3,2)$ the square of the horizontal and vertical parts thus have the same sign]. The square of the total spin is then related to the square of the vertical part.

We come now to consider more general solutions of the Einstein equations in r dimensions. We contemplate only the case where the topology of the space G remains unchanged. The structure of the principal fibre bundle P is also kept unaltered to ensure the uniqueness of the projection on space-time of the new metric on G (denoted also by γ). This means we must still have $r-k$ Killing vector fields B_M which commute like the base vectors of the Lie algebra of H,

$$[B_M, B_N] = c^P{}_{MN} B_p \tag{4}$$

The B_M satisfy $B_M(\gamma) = 0$.

A local coordinate system can again be introduced in which only the last $r-k$ components of B_M are different from zero. The B_M form again the vertical vector space of P which is labeled by the last $r-k$ coordinates. The new metric γ determines a modified horizontal vector space which is orthogonal to the vertical vectors B_M. One can introduce an orthonormal base B_E. The B_E and B_M are in the case of the de Sitter metric an orthonormal base of left invariant vectors of G belonging to $\mathfrak{k}$ and $\mathfrak{h}$, respectively. Now the commutation relations among the B_E are not prescribed, but they obey the relations

$$[B_M, B_E] = c^F{}_{ME} B_F \tag{4a}$$

and thus fulfill the Killing conditions. The projection of the new γ on the base manifold, which we denote henceforth by B, gives rise to a modified general Riemannian metric g on B. We have not allowed alterations of the topology of G and we can therefore contemplate the connection with the base of left invariant horizontal vectors A_E and vertical vectors A_M simultaneously with the new connection (and metric), which has a base of horizontal vectors B_E and vertical vectors B_M. We can even set $A_M = B_M$ and impose the boundary condition that in the limit of increasing spatial distance from a localized source the new metric γ should tend toward the Cartan-Killing metric and $B_E \to A_E$. The new metric and connections are adapted to the physical conditions; the original one serves only for the boundary conditions, as in the bimetric theories of space-time.

An arbitrary connection of the linear frame bundle of B describes a linear connection on B which is metric, i.e., the metric g on B is covariant constant with respect to the linear connection. The connection may, however, have nontrivial torsion if the distribution of horizontal subspaces on G differ from that given by the A_E. The metric g can be a general Lorentz metric. The degrees of freedom of the metric γ correspond on B to the degrees of freedom of the metric g and of torsion or to the metric and linear connection on B.

The solutions of Einstein's equations must obey the subsidiary conditions imposed by the choice of the B_M. This can either be achieved by Lagrangian multipliers or by assuming the components γ_{mn} of the metric (respectively the vertical-vertical components of the metric or of the vector fields B_M) as determined entities in the equations and solving for the remaining components as unknown variables.

We can thus choose as the Lagrangian density in r dimensions the sum of a metric part and a matter part:

$$\mathscr{L}^{(r)} = \mathscr{L}_G^{(r)} + \mathscr{L}_M^{(r)} = \gamma^{1/2}(R - 2 + L_M) \tag{5}$$

Both parts of $\mathscr{L}^{(r)}$ must fulfill the Killing conditions, which in suitable coordinates, where the x^m determine only the points of the fibre, simply state that $\mathscr{L}^{(r)}$ does not depend on the x^m. To achieve this, one can vary all the γ and insert the determined components into the field equations.

One can also express $\mathscr{L}^{(r)}$ exclusively in terms of entities defined on the base manifold. $\mathscr{L}_G$ may thus be expressed in terms of the metric g and the torsion tensor, or, as it often proves more suitable, in terms of Yang-Mills potentials which are related to the linear connection and, instead of the metric g, in terms of a system of tetrades B_E^i which may be considered as the projections of the B_E. The dependence on the points on the fibres corresponds then to the gauge and the choice of the tetrade frame on B. The matter Lagrangian $\mathscr{L}_M$ may even require the introduction of the tetrades instead of the metric g (Rosenfeld, 1940).[5]

The projection of geodesics on the base does not in the generalized case always describe the motion of a spinning test particle correctly. This is not surprising, since a point particle cannot fulfill the Killing condition of $\mathscr{L}_G$. We shall have to consider sources that extend over the fibres—or equivalently gauge-convariant sources.

The vertical-vertical part of Einstein's equations are not obtained if the Lagrangian on G is expressed as a Lagrangian on B in terms of tetrades and a gauge potential only. One could ask whether they should remain in the theory at all, considering that the remaining equations may be interpreted as a consistent Einstein-Yang-Mills theory. Our point of view is that the theory should be constructed as close to the geometry of the de Sitter group as possible and also these equations must be considered. We shall see that

[5]Gravitational theories with a gauge field of the tetrade rotations have been considered much earlier by the author in order to make Moeller's energy-momentum complex unique and explain PC violation (Halpern and Miketinac, 1970) and in connection with a unified invariance group of the Dirac equations (Halpern, 1977a). The unique properties of the present theory are prescribed by the topology of the de Sitter group.

they impose relations between the Yang–Mills fields and the sources which supplement the boundary conditions. This may even be required because H is noncompact with an indefinite metric.

The Lie algebra valued connection one-form ω can be expressed in a base B^R dual to the B_R as

$$\omega = B^M A_M, \qquad A_M \in \mathfrak{h} \tag{6}$$

Its curvature two-form Ω is

$$\Omega = d\omega + [\omega, \omega] \tag{6a}$$

or in a coordinate frame

$$\Omega_{ik} = F^M{}_{ik} A_M = [B^M{}_{k,i} - B^M{}_{i,k} + c^M{}_{PQ} B^P{}_i B^Q{}_k] A_M \tag{6a$'$}$$

The corresponding linear connection in an orthonormal frame has the components

$$\Gamma^J{}_{EF} = -B^M{}_k b^k{}_E c^J{}_{FM} \tag{6b}$$

with $b_E = \alpha^{-1}\pi' B_E$ the orthonormal base vectors on B induced by B_E. The $\mathbb{R}^k$ valued soldering form θ gives the horizontal components of a vector on P with respect to the base B_E:

$$\theta = B^E \qquad (E = 1, \ldots, k) \tag{7}$$

The torsion two-form is

$$F^E = d\theta + \tilde{\omega} \wedge \theta \tag{7a}$$

with the connection of the frame bundle $\tilde{\omega}$. The components

$$F^E{}_{ik} = B^E{}_{k,i} - B^E{}_{i,k} - c^E{}_{FM}[B^M{}_i B^F{}_k - B^M{}_k B^F{}_i] \tag{7a$'$}$$

In a coordinate frame the torsion tensor is then defined as

$$f^e{}_{ik} = \alpha^{-1} b^e{}_E F^E{}_{ik} \tag{7b}$$

and the linear connection on B:

$$\Gamma^e{}_{ik} = \left\{ {e \atop i \;\; k} \right\} + K^e{}_{ik}, \qquad K^e{}_{ik} = \tfrac{1}{2}[f^e{}_{\cdot\, ik} + f_i{}^e{}_{\cdot\, k} + f_k{}^e{}_{\cdot\, i}] \tag{7c}$$

is the sum of the Christoffel connection and the contortion tensor K.

The components of the curvature tensor of Γ are related to the curvature two-form:

$$c^E{}_{FM} F^M{}_{ij} B^i{}_I B^j{}_J = c^E{}_{FM} F^M{}_{IJ} = \alpha^2 f^E{}_{FIJ}$$

$$\begin{aligned} f^E{}_{FIJ} = {} & \Gamma^E{}_{IF|J} - \Gamma^E{}_{JF|I} + \Gamma^E{}_{JD}\Gamma^D{}_{IF} - \Gamma^E{}_{ID}\Gamma^D{}_{JF} \\ & + \Gamma^E{}_{HF}[\Gamma^H{}_{IJ} - \Gamma^H{}_{JI} - f^H{}_{IJ}] \end{aligned} \tag{6c}$$

$B^M{}_i$ can be related to the components of a Yang–Mills potential and the transformations between different local trivialisation of P to the gauge transformations.

The transition to a different local trivialisation of P is equivalent to a coordinate transformation in r dimensions of the special form

$$x'^k = x^k, \qquad x'^m = \phi^m(x^k, a^n(x^n)) \tag{8}$$

where the ϕ are the composition functions of the elements of the group G. On the base manifold this special set of coordinate transformations results in a tetrade rotation with the (point-dependent) element a^n of H, associated with a gauge transformation of the "Yang–Mills potentials" B^M with the same element of H; the latter undergo an inhomogeneous transformation due to the dependence of a^n on the point of B, whereas the tetrade transformation is homogeneous.

The Langrangian $\mathscr{L}_G^{(r)}$ [Eq. (5)], which does not depend on x^m, is equivalent to the Lagrangian $\mathscr{L}_G^{(k)}$ on the base manifold:

$$\mathscr{L}_G^{(k)} = g^{1/2}(R^{(k)} - \alpha^{-2} + \tfrac{1}{4}\alpha^2 f^M{}_{ik} f^{Nik} \gamma_{MN}) \tag{5a}$$

where $f^M{}_{ik} = F^M{}_{ik}$, but its indices are now raised with the metric g instead of the metric γ. The value of the cosmological constant is changed due to a constant term formed out of the B_M. The vertical-vertical part of the original equation has to be considered as subsidiary conditions for the solutions.

4. THE FIELD EQUATIONS

The Einstein equations in r dimensions in cosmological units with a matter source τ as right-hand member are obtained by variation of the Langrangian $\mathscr{L}^{(r)}$ with respect to the metric γ:

$$R_{uv} - \tfrac{1}{2}\gamma_{uv}(R-2) = \tau_{uv} \tag{9}$$

We write them first in an orthonormal frame, in which the horizontal, mixed, and vertical components can be distinguished and express their left side in terms of the metric g and the Yang–Mills fields f. We have now to distinguish between the Ricci tensor in r and in k dimensions in the different parts of these equations. We no longer add labels to achieve this and use the same conventional symbol R_{EF} for both the horizontal components of the tensor in r dimensions as well as for the tensor on the base manifold. The two entities are easily distinguishable because of the use of the metric g and the dimensional constant α on the base. The entities at the base pertain to a system of the tetrades b_E, which are the projection of the B_E from a point on each fibre determined by the gauge of the Yang–Mills fields

f^M. We have

$$R_{EH}-\tfrac{1}{2}\gamma_{EH}(R-2)$$
$$=\tau_{EH}=\alpha^2[R_{EH}+\tfrac{1}{4}f^M{}_{JE}f_M{}^J{}_H-\tfrac{1}{2}g_{EH}(R-\alpha^{-2}+\tfrac{1}{4}\alpha^2 f^M{}_{ik}f_M{}^{ik})] \tag{9a}$$

$$R_{NE}=\tfrac{1}{2}\gamma_{MN}\alpha f^{MH}{}_{E\|H}=\tau_{NE} \tag{9b}$$

$$R_{NP}-\tfrac{1}{2}\gamma_{NP}(R-2)$$
$$=\tau_{NP}=\alpha^2[-\tfrac{1}{4}\alpha^2 f_{Nik}f_p{}^{ik}-\tfrac{1}{2}\gamma_{NP}(R+\tfrac{1}{4}\alpha^2 f^M{}_{ik}f_M{}^{ik}-\tfrac{5}{3}\alpha^{-2})] \tag{9c}$$

Notice that g^{ik} and b^i are not dimensionless, yet $g_{EH}=\gamma_{EH}$ is. In cosmological units of length $\alpha=1$. The double bar in the second set of equations denotes an invariant derivative of the Yang–Mills field.

The tensors on both sides of these equations must have vanishing Lie derivatives with respect to the A_M and a vanishing covariant divergence. The vanishing of the Lie derivatives determines the transformation properties of the components with respect to the combined gauge and tetrade transformations. The covariant conservation in r dimensions leads in k dimensions to the covariant conservation of the spin current and the fact that in general only the energy-momentum tensor τ of the matter field and that of the Yang–Mills field together are covariantly conserved,

$$(g^{1/2}T^k{}_h)_{;k}=\alpha^{-2}f^M{}_{hk}g^{1/2}j^k{}_M \tag{10a}$$

$$(g^{1/2}j^k{}_N)_{,k}=j^k{}_N c^N{}_{MP}B^P{}_k \tag{10b}$$

with

$$g^{1/2}T^{ik}=\delta\mathcal{L}^{(k)}/\delta g_{ik}, \qquad g^{1/2}j^k{}_M=\delta\mathcal{L}^{(k)}/\delta B^M{}_K$$

These equations relate the equations of motion of sources to the field equations.

The boundary conditions exclude the vanishing of the Yang–Mills fields. The indefinite metric of the noncompact group H allows, however, the vanishing of their total contribution in the energy-momentum tensor. They can be of cosmological smallness in the absence of sources (e.g., the de Sitter universe where $f^M{}_{ik}=-c^M{}_{EF}A^E A^F$ everywhere).

The Yang–Mills fields f^M are expressible in terms of the tetrades and the contortion, out of which the fundamental geometrical entities of the metric tensor and the torsion tensor are constructed [see equation (7c)]. Variation of the Lagrangian $\mathcal{L}^{(k)}$ with respect to these variables results in the alternative form of the field equations:

$$\frac{\delta\mathcal{L}_G^{(k)}}{\delta b^k{}_A}b_i^E\gamma_{EA}=g^{1/2}[R_{ik}-\tfrac{1}{2}g_{ik}(R-\alpha^{-2}+\frac{\alpha^2}{4}f^M{}_{jh}f_M{}^{jh})$$
$$+\frac{\alpha^2}{2}f^M{}_{ih}f_{Mk}{}^h+\bar{\tau}_{ik}]=g^{1/2}T_{ik} \tag{11a}$$

$$\bar{\tau}^{ik} = 3[j^{[i,h]k} + j^{[k,h]i} + j^{[k,i]h}]_{;h} \tag{11b}$$

with

$$j^{[ik]h} = j^{h}{}_{N} c^{NIK} b^{i}{}_{I} b^{k}{}_{K}$$

The term $\bar{\tau}_{ik}$ arises from the variation of the Yang–Mills part with respect to the tetrades.

$\bar{\tau}$ and other additional terms in the field equations relate to a part of the Yang–Mills Lagrangian which corresponds to a term quadratic in the Riemann tensor and to the coupling terms of the metric with the contortion K,

$$\mathscr{L}_G^{(k)} = g^{1/2}(R - \alpha^{-2} + \tfrac{3}{4}\alpha^2 f_{ijhk} f^{ijhk}) \tag{12}$$

$$f^{i}{}_{jhk} = \{-R^{i}{}_{jhk} + K^{i}{}_{hj;k} - K^{i}{}_{kj;h} + K^{i}{}_{kd} K^{d}{}_{hj} - K^{i}{}_{hd} K^{d}{}_{kj}\} \tag{12a}$$

$$\frac{\delta \mathscr{L}_G}{\delta K^{a}{}_{bc}} = \frac{\delta \mathscr{L}_G}{\delta B^{M}{}_{b}}(-3 c^{MD}{}_{A} b^{c}{}_{D} b^{A}{}_{a}) = -3 j^{[c,a]b} \tag{13}$$

In analogy to the relations

$$j^{k}{}_{M} = \alpha^2 (g^{1/2} f_M{}^{kh})_{\|h}, \qquad j^{k}{}_{M\|k} = 0 \tag{14}$$

we can write

$$j^{[c,a]k} = \alpha^2 \nabla_h f^{cakh}, \qquad \nabla_k j^{[c,a]h} = 0 \tag{14a}$$

where ∇_h denotes the mixed covariant derivative, which acts on the first two indices a and c with the connection Γ and on the remaining indices with the Christoffel connection. We thus find

$$j^{[c,a]k} = \alpha^2 \{-R^{cakh}{}_{;h} - K^{cka}{}_{;h;}{}^{h} + K^{cha}{}_{;}{}^{k}{}_{;h} + (K^{ck}{}_{d} K^{dha} - K^{ch}{}_{d} K^{dha})_{;h}$$
$$+ f^{dakh} K^{c}{}_{hd} + f^{adkh} K^{c}{}_{hd}\} \tag{15}$$

The first term with the Riemann tensor on the right due to the Bianchi identities equals

$$-R^{cakh}{}_{;h} = R^{ck}{}_{;}{}^{a} - R^{ak}{}_{;}{}^{c} \tag{16}$$

The matter part of the Lagrangian $\mathscr{L}_M^{(r)}$ is constructed out of wave functions ϕ^{ξ} and their derivatives and of the metric γ. Totally, it must be independent of the x^m in any local trivialization; however, the ϕ^{ξ} are in general scalar functions of all the variables x^r. They should in case of the natural metric γ on P form such realizations of G that the total independence of $\mathscr{L}_M$ on the x^m arises. In general one will also demand that the ϕ^{ξ} are eigenstates of the Casimir operator of G. They may be eigenfunctions of other operators of G. In principle all the eigenvalues that occur in representations of G or its subgroups have local eigenfunctions on the group manifold. This is even true for half-integer eigenvalues of generators of the rotation group (Bopp and Haag, 1950).

The four-dimensional form of this Lagrangian on the base manifold is again obtained by expressing it in terms of tetrade and Yang–Mills fields or torsion. Horizontal derivatives become covariant derivatives with connection Γ and vertical derivatives correspond to generators of the realization of H; they give rise to the source of the vertical part of Einstein's equations.

5. ASPECTS OF PHYSICAL INTERPRETATION

The present Kaluza–Klein theory on the manifold of the de Sitter group provides us with a habitat in the form of the de Sitter universe; it differs, among other ways, from those Kaluza–Klein approaches, which may now be termed conventional, in the aspect of a cosmological member of the right magnitude α^{-2}. The occurrence of a nonlinear part of the gravitational Lagrangian with the giant factor α^2 results, however, inevitably from these features within the scope of the ten-dimensional theory. The boundary conditions at large distances from an inhomogeneous matter distribution require that this term vanish there and in many cases the symmetry is such that it gives negligible contributions everywhere.

The vacuum solutions of a metric theory without torsion are not affected by such a term. The presence of matter causes, however, difficulties in this special case. Such difficulties may, however, disappear when torsion is taken into account. The vertical part of the equations requires indeed that the curvature tensor f out of which the nonlinear term in the Lagrangian is constructed should be of cosmological smallness everywhere except at locations where matter is present and even there it should not be larger than the contribution from the source. The Riemann tensor, which is a constituent of f, has no such restriction, so that the condition can only be fulfilled with a compensating term of the contortion tensor K.

A large torsion field of this kind must, however, differ in its properties considerably from the conventional fields of physics. We first consider the justified distrust that the physicist may show against such a newcomer and consider the case where the torsion occurs only as a minor disturbance, so that the term resulting from the nonlinear part of the Lagrangian $\mathscr{L}_G^{(k)}$ dominates the transverse field equations due to the factor α^2. We must thus forget about the vertical part of the fields equations, which acts in any case only as subsidiary conditions. A term that may dominate due to the factor α^2 if suitable sources are present is the one expressed in Section 4 by the covariant derivatives of the current j [equation (11b)].

The orbital angular momentum due to a localized weak source in the neighborhood of a given point can be determined approximately for this term and it can be shown that under our assumptions it is equal to the current j itself. Thus $g^{1/2}(x^i T^{0k} - x^k T^{0i})$ is equivalent in our limit to $j^{[i,k]0}$. The explicit expression for j given in Section 4 shows that even if j vanishes, a source of the torsion fields expressible in terms of covariant derivatives of the Ricci tensor remains [equation (15)].

The presence of torsion thus cannot in general be excluded. This may bring us closer to the interpretation imposed by the vertical part of the equations: The contribution of the torsion is comparable in absolute value to that of the metric and directed such that the curvature tensor f remains small.

The theory thus yields a metric which is rather close to that of general relativity—in particular if massive sources are absent. Large torsion fields exist, but their contribution to the physics is not significant. The physical quantity that counts for effects beyond those of general relativity is the curvature tensor f, which is that of the Yang–Mills field related to the metric by subsidiary conditions from the vertical equations. The metric and torsion tensors, which are well-established geometrical entities, do not enter here independently into the physical laws. Physical predictions beyond this recognition must be regarded still with the greatest caution because of the lack of anything more than estimates of solutions for the field equations and even more so for the sources.

Elementary particles with spin should perform a motion in vacuum which, due to the smallness of the tensor f compared to the Riemann tensor (the compensating effect of torsion), will deviate correspondingly less from the geodesic orbits than is predicted by Papapetrou's (1951) equations. More difficult in view of the absence of a precise model are predictions about the direction of the spin axis of orbiting elementary particles. There can, however, be no doubt that also for this case f and not the Riemann tensor is the pertinent quantity to determine the effect. The axis of an elementary particle gyroscope should thus deviate much less from the fixed stars than according to general relativity. Observation of such effects may become possible through comparison of the state of polarization and direction of light of complete circular polarization after its deflection by a massive source. However, I would prefer to consider such effects first in a higher dimensional theory of this kind where both spin and charge are inner degrees of freedom described in a unified geometrical theory.

I see an important criterion for a correct physical theory of this kind in the equivalence of the interaction of spin and orbital angular momentum. The complex structure of orbiting systems and their binding forces, which is not even solved for the two-body system in relativity, makes it still impossible to prove such a conjecture in the present context. The fact that here spinning bodies are not the only sources of torsion makes the assumption, however, a lot more justifiable. The Stanford gyroscope experiment would thus be suited to obtain a decision on the physical correctness of the theory.

6. REMARKS ABOUT THE EXTENSION TO THE CONFORMAL GROUPS

The formalism developed here can be applied to the 15-dimensional conformal groups for G with a 10-dimensional de Sitter group for H. The base manifold B in this case is a space of constant curvature with $n(n+1)/2$ Killing vector fields. The manifold B can thus be considered as the space of a Kaluza-Klein theory on which a generalized metric is introduced with one remaining Killing vector field, which determines the projection on a four-dimensional factor space which is to be the space-time manifold.

One has to consider for G the conformal group $SO(4, 2)$ or the anticonformal group $SO(3, 3)$, and for H either $SO(4, 1)$ or $SO(3, 2)$. One gets correspondingly for G/H spaces of constant curvature which are spatially closed and are open in two timelike dimensions, or alternatively four open space dimensions and a closed timelike direction, or, in the last case, a spatially open space which is closed in the two timelike dimensions. The group $SO(3, 3)$ is not so frequently considered as $SO(4, 2)$, but it seems to play a fundamental role in the present context. If one tries to unify the groups of spin transformations and electromagnetic gauge transformtions into one larger semisimple group (Halpern, 1977a, one arrives at a group which is locally isomorphic to $SO(3, 3)$ and to $SL(4r)$ and not to $SO(4, 2)$. This group contains the transformations of the two-dimensional vector space of complex numbers and of two-dimensional spinors. A formal geometric unification on the 15-dimensional group manifold of the present theory with electrodynamics is not difficult to achieve, but one can hope to find in such a generalization a geometrical description of the relation of the fundamental length which occurs here and the Planck length, which, according to the ideas of Dirac, Einstein, and Jordan, may vary with time. Such problems will be discussed in detail in a subsequent publication.

ACKNOWLEDGMENTS

I am indebted to Prof. R. Jantzen of the Department of Mathematics of Villanova University for his frequent helpful discussions on the mathematical structure of the theory and for its reformulation in sound mathematical lenguage.

I thank Prof. B. Laurent of the University of Stockholm for stimulating discussions on the physical foundations of the theory and Dr. M. Bozic of the Institute of Physics, Beogard, for an important suggestion on the representation of half-integer angular momentum on the group manifold. I thank Prof. T. Damour, Meudon, for numerically significant dimensional considerations and Dr. J. Åman, Stockholm, for the examination of various properties of metrics with the powerful computer program Sheep.

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